Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Complex Numbers P3

Q1

The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C. [4]

(ii) Express $\frac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\frac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

Q2

The complex number u is given by

$$u = \frac{3 + \mathrm{i}}{2 - \mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

Q3

The complex number $\frac{2}{-1 + \mathrm{i}}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

Q4

(a) The complex number z is given by $z = \frac{4 - 3\mathrm{i}}{1 - 2\mathrm{i}}$.

(i) Express z in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Find the modulus and argument of z. [2]

(b) Find the two square roots of the complex number $5 - 12\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

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Q5

The variable complex number z is given by

$$z = 2\cos\theta + \mathrm{i}(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - \mathrm{i}| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\dfrac{1}{z + 2 - \mathrm{i}}$ is constant for $-\pi < \theta < \pi$. [4]

Q6

The complex number w is given by $w = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\dfrac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\dfrac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2\mathrm{i}$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

Q7

(i) Solve the equation $z^2 + (2\sqrt{3})\mathrm{i}z - 4 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

Q8

The complex number $-2 + \mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of u. [2]

(iv) Sketch an Argand diagram showing the point representing u. Shade the region whose points represent the complex numbers z satisfying both the inequalities

$$|z| < |z - 2| \quad \text{and} \quad 0 < \arg(z - u) < \tfrac{1}{4}\pi. \qquad [4]$$

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Q9

The complex numbers $-2 + \mathrm{i}$ and $3 + \mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, the complex numbers

(a) $u + v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

Q10

The variable complex number z is given by

$$z = 1 + \cos 2\theta + \mathrm{i} \sin 2\theta,$$

where θ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of z is $2\cos\theta$ and the argument of z is θ. [6]

(ii) Prove that the real part of $\frac{1}{z}$ is constant. [3]

Q11

The complex number $2 + 2\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - \mathrm{i}|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

Q12

(a) The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + \mathrm{i}\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + \mathrm{i}\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1 - \mathrm{i}\sqrt{3}| \leqslant 1$ and $\arg z \leqslant \frac{1}{3}\pi$. [5]

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Associate *of* Royal College of Science, UK
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Facebook/Learning Math Online

Q13

The complex number z is given by

$$z = (\sqrt{3}) + \mathrm{i}.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x + \mathrm{i}y$, where x and y are real,

(a) $2z + z^*$,

(b) $\dfrac{\mathrm{i}z^*}{z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and $\mathrm{i}z^*$ respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

Q14

The complex number w is defined by $w = 2 + \mathrm{i}$.

(i) Showing your working, express w^2 in the form $x + \mathrm{i}y$, where x and y are real. Find the modulus of w^2. [3]

(ii) Shade on an Argand diagram the region whose points represent the complex numbers z which satisfy

$$|z - w^2| \leqslant |w^2|.$$ [3]

Q15

The complex number u is defined by $u = \dfrac{6 - 3\mathrm{i}}{1 + 2\mathrm{i}}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z - u) = \frac{1}{4}\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z - (1 + \mathrm{i})u| = 1$, find the greatest possible value of $|z|$. [3]

Q16

(a) The complex number u is defined by $u = \dfrac{5}{a + 2\mathrm{i}}$, where the constant a is real.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Find the value of a for which $\arg(u^*) = \frac{3}{4}\pi$, where u^* denotes the complex conjugate of u. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z| < |z - 2 - 2\mathrm{i}|$. [4]

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Q17

(i) Find the roots of the equation

$$z^2 + (2\sqrt{3})z + 4 = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [2]

(ii) State the modulus and argument of each root. [3]

(iii) Showing all your working, verify that each root also satisfies the equation

$$z^6 = -64.$$ [3]

Q18

(a) Showing your working, find the two square roots of the complex number $1 - (2\sqrt{6})i$. Give your answers in the form $x + iy$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3i| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

Q19

The complex number w is defined by $w = -1 + i$.

(i) Find the modulus and argument of w^2 and w^3, showing your working. [4]

(ii) The points in an Argand diagram representing w and w^2 are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z - (a + bi)| = k$. [4]

Q20

The complex number u is defined by $u = \dfrac{(1 + 2i)^2}{2 + i}$.

(i) Without using a calculator and showing your working, express u in the form $x + iy$, where x and y are real. [4]

(ii) Sketch an Argand diagram showing the locus of the complex number z such that $|z - u| = |u|$. [3]

Q21

Throughout this question the use of a calculator is not permitted.

The complex number u is defined by

$$u = \frac{1 + 2i}{1 - 3i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Show on a sketch of an Argand diagram the points A, B and C representing the complex numbers u, $1 + 2i$ and $1 - 3i$ respectively. [2]

(iii) By considering the arguments of $1 + 2i$ and $1 - 3i$, show that

$$\tan^{-1}2 + \tan^{-1}3 = \tfrac{3}{4}\pi.$$ [3]

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Q22

(a) The complex numbers u and w satisfy the equations

$$u - w = 4\mathrm{i} \quad \text{and} \quad uw = 5.$$

Solve the equations for u and w, giving all answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 + 2\mathrm{i}| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of z. [5]

(ii) Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

Q23

The complex number $1 + (\sqrt{2})\mathrm{i}$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by $\mathrm{p}(x)$.

(i) Showing your working, verify that u is a root of the equation $\mathrm{p}(x) = 0$, and write down a second complex root of the equation. [4]

(ii) Find the other two roots of the equation $\mathrm{p}(x) = 0$. [6]

Q24

(a) Without using a calculator, solve the equation $\mathrm{i}w^2 = (2 - 2\mathrm{i})^2$. [3]

(b) **(i)** Sketch an Argand diagram showing the region R consisting of points representing the complex numbers z where

$$|z - 4 - 4\mathrm{i}| \leqslant 2.$$ [2]

(ii) For the complex numbers represented by points in the region R, it is given that

$$p \leqslant |z| \leqslant q \quad \text{and} \quad \alpha \leqslant \arg z \leqslant \beta.$$

Find the values of p, q, α and β, giving your answers correct to 3 significant figures. [6]

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Answers:

Q1:

(i) Show u and u^* in relatively correct positions
Show $u + u^*$ in relatively correct position
State or imply that $OACB$ is a parallelogram
State or imply that $OACB$ has a pair of adjacent equal sides
[The statement that $OACB$ is a rhombus, or equivalent, earns B2✓.]

(ii) *EITHER*: Multiply numerator and denominator of $\frac{u}{u^*}$ by $2 + \mathrm{i}$
Simplify numerator to $3 + 4\mathrm{i}$ or denominator to 5
Obtain answer $\frac{3}{5} + \frac{4}{5}\mathrm{i}$, or equivalent
OR: Obtain two equations in x and y, and solve for x or for y
Obtain $x = \frac{3}{5}$ or $y = \frac{4}{5}$
Obtain answer $\frac{3}{5} + \frac{4}{5}\mathrm{i}$

(iii) *EITHER*: State or imply $\arg\left(\frac{u}{u^*}\right) = 2\arg u$
Justify the given statement correctly
OR: Use tan 2*A* formula with $\tan A = \frac{1}{2}$
Justify the given statement correctly
[The f.t. is on $-2 + \mathrm{i}$ as complex conjugate.]

Q2:

(i) *EITHER*: Multiply numerator and denominator by $2 + \mathrm{i}$, or equivalent
Simplify numerator to $5 + 5\mathrm{i}$ or denominator to 5
Obtain answer $1 + \mathrm{i}$
OR: Obtain two equations in x and y, and solve for x or for y
Obtain $x = 1$
Obtain $y = 1$
OR: Using correct processes express u in polar form
Obtain $u = \sqrt{2}\,(\cos 45° + \mathrm{i}\sin 45°)$, or equivalent
Obtain answer $1 + \mathrm{i}$

(ii) State that the modulus is $\sqrt{2}$ or 1.41
State that the argument is 45° or $\frac{1}{4}\pi$ (or 0.785)

(iii) Show the point representing u in a relatively correct position
Show a circle with centre at the point representing u
Indicate or imply the radius is 1
[NB: If the Argand diagram has unequal scales the locus is not circular in appearance, but an ellipse with centre u and equal axes parallel to the axes of the diagram earns B1✓, and B1 if both semi-axes are indicated or implied to be equal to 1. In such a situation only award B1✓ for a circle with centre u and a horizontal or vertical radius indicated or implied to be 1.]

(iv) Carry out complete strategy for calculating min$|z|$ for the locus
Obtain answer $\sqrt{2} - 1$ (or 0.414)
[The f.t. is on the value of u.]

Q3:

(i) *EITHER*: Carry out multiplication of numerator and denominator by $-1 - \mathrm{i}$, or solve for x or y
Obtain $u = -1 - \mathrm{i}$, or any equivalent of the form $(a + \mathrm{i}b)/c$
State modulus of u is $\sqrt{2}$ or 1.41
State argument of u is $-\frac{3}{4}\pi$ (-2.36) or $-135°$, or $\frac{5}{4}\pi$ (3.93) or 225°
OR: Divide the modulus of the numerator by that of the denominator
State modulus of u is $\sqrt{2}$ or 1.41
Subtract the argument of the denominator from that of the numerator, or equivalent
State argument of u is $-\frac{3}{4}\pi$ (-2.36) or $-135°$, or $\frac{5}{4}\pi$ (3.93) or 225°
Carry out method for finding the modulus or the argument of u^2
State modulus of u is 2 and argument of u^2 is $\frac{1}{2}\pi$ (1.57) or 90°

(ii) Show u and u^2 in relatively correct positions
Show a circle with centre at the origin and radius 2
Show the line which is the perpendicular bisector of the line joining u and u^2
Shade the correct region, having obtained u and u^2 correctly

Q4:

(a) **(i)** *EITHER*: Carry out multiplication of numerator and denominator by $1 + 2\mathrm{i}$, or equivalent
Obtain answer $2 + \mathrm{i}$, or any equivalent of the form $(a + \mathrm{i}b)/c$
*OR*1: Obtain two equations in x and y, and solve for x or for y
Obtain answer $2 + \mathrm{i}$, or equivalent
*OR*2: Using the correct processes express z in polar form
Obtain answer $2 + \mathrm{i}$, or equivalent

(ii) State that the modulus of z is $\sqrt{5}$ or 2.24
State that the argument of z is 0.464 or 26.6°

(b) *EITHER*: Square $x + \mathrm{i}y$ and equate real and imaginary parts to 5 and -12 respectively
Obtain $x^2 - y^2 = 5$ and $2xy = -12$
Eliminate one variable and obtain an equation in the other
Obtain $x^4 - 5x^2 - 36 = 0$ or $y^4 + 5y^2 - 36 = 0$, or 3-term equivalent
Obtain answer $3 - 2\mathrm{i}$
Obtain second answer $-3 + 2\mathrm{i}$ and no others
[SR: Allow a solution with $2xy = 12$ to earn the second A1 and thus a maximum of 3/6.]
OR: Convert $5 - 12\mathrm{i}$ to polar form (R, θ)
Use the fact that a square root has the polar form $(\sqrt{R}, \frac{1}{2}\theta)$
Obtain one root in polar form, e.g. $(\sqrt{13}, -0.588)$ or $(\sqrt{13}, -33.7°)$
Obtain answer $3 - 2\mathrm{i}$
Obtain answer $-3 + 2\mathrm{i}$ and no others

Q5:

(i) Find modulus of $2\cos\theta - 2\mathrm{i}\sin\theta$ and show it is equal to 2
Show a circle with centre at the point representing **i**
Show a circle with radius 2

(ii) Substitute for z and multiply numerator and denominator by the conjugate of $z + 2 - \mathrm{i}$, or equivalent
Obtain correct real denominator in any form
Identify and obtain correct unsimplified real part in terms of $\cos\theta$, e.g. $(2\cos\theta + 2)/(8\cos\theta + 8)$
State that real part equals $\frac{1}{4}$

Q6:

(i) State that the modulus of w is 1
State that the argument of w is $\frac{2}{3}\pi$ or 120° (accept 2.09, or 2.1)

(ii) State that the modulus of wz is R
State that the argument of wz is $\theta + \frac{2}{3}\pi$
State that the modulus of z/w is R
State that the argument of z/w is $\theta - \frac{2}{3}\pi$

(iii) State or imply the points are equidistant from the origin
State or imply that two pairs of points subtend $\frac{2}{3}\pi$ at the origin, or that all three pairs subtend equal angles at the origin

(iv) Multiply $4 + 2\mathrm{i}$ by w and use $\mathrm{i}^2 = -1$
Obtain $-(2 + \sqrt{3}) + (2\sqrt{3} - 1)\mathrm{i}$, or exact equivalent
Divide $4 + 2\mathrm{i}$ by w, multiplying numerator and denominator by the conjugate of w, or equivalent
Obtain $-(2 - \sqrt{3}) - (2\sqrt{3} + 1)\mathrm{i}$, or exact equivalent
[Use of polar form of $4 + 2\mathrm{i}$ can earn M marks and then A marks for obtaining exact $x + \mathrm{i}y$ answers.]
[SR: If answers only seen in polar form, allow B1+B1 in (i), B1√ + B1√ in (ii), but A0 + A0 in (iv).]

Fayzan Munawar
Associate *of* Royal College of Science, UK
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Facebook/Learning Math Online

Q7:

(i) Use quadratic formula, or completing the square, or the substitution $z = x + iy$ to find a root, using $i^2 = -1$
Obtain a root, e.g. $1 - \sqrt{3}i$
Obtain the other root, e.g. $-1 - \sqrt{3}i$

(ii) Represent both roots on an Argand diagram in relatively correct positions

(iii) State modulus of both roots is 2
State argument of $1 - \sqrt{3}i$ is $-60°$ (or $300°$, $-\frac{1}{3}\pi$, $-\frac{5}{3}\pi$)
State argument of $-1 - \sqrt{3}i$ is $-120°$ (or $240°$, $-\frac{2}{3}\pi$, $-\frac{4}{3}\pi$)

(iv) Give a complete justification of the statement
[The A marks in **(i)** are for the final versions of the roots. Allow $(\pm 2 - 2\sqrt{3}i)/2$ as final answer. The remaining marks are only available for roots such that $xy \neq 0$.]
[Treat answers to **(iii)** in polar form as a misread]

Q8:

(i) Substitute $x = -2 + i$ in the equation and attempt expansion of $(-2 + i)^3$
Use $i^2 = -1$ correctly at least once and solve for k
Obtain $k = 20$

(ii) State that the other complex root is $-2 - i$

(iii) Obtain modulus $\sqrt{5}$
Obtain argument 153.4° or 2.68 radians

(iv) Show point representing u in relatively correct position in an Argand diagram
Show vertical line through $z = 1$
Show the correct half-lines from u of gradient zero and 1
Shade the relevant region
[SR: For parts **(i)** and **(ii)** allow the following alternative method:
State that the other complex root is $-2 - i$
State quadratic factor $x^2 + 4x + 5$
Divide cubic by 3-term quadratic, equate remainder to zero and solve for k, or, using 3-term quadratic, factorise cubic and obtain k
Obtain $k = 20$

Q9:

(i) **(a)** State that $u + v$ is equal to $1 + 2i$

(b) *EITHER*: Multiply numerator and denominator of u/v by $3 - i$, or equivalent
Simplify numerator to $-5 + 5i$, or denominator to 10
Obtain answer $-\frac{1}{2} + \frac{1}{2}i$, or equivalent
OR1: Obtain two equations in x and y and solve for x or for y
Obtain $x = -\frac{1}{2}$ or $y = \frac{1}{2}$
Obtain answer $-\frac{1}{2} + \frac{1}{2}i$, or equivalent
OR2: Using the correct processes express u/v in polar form
Obtain $x = -\frac{1}{2}$ or $y = \frac{1}{2}$ correctly
Obtain answer $-\frac{1}{2} + \frac{1}{2}i$, or equivalent

(ii) State that the argument of u/v is $\frac{3}{4}\pi$ (2.36 radians or 135°)

(iii) *EITHER*: Use facts that angle $AOB = \arg u - \arg v$ and $\arg u - \arg v = \arg(u/v)$
Obtain given answer
OR1: Obtain $\tan A\hat{O}B$ from gradients of OA and OB and the $\tan(A \pm B)$ formula
Obtain given answer
OR2: Obtain $\cos A\hat{O}B$ by using the cosine formula or scalar product
Obtain given answer

(iv) State $OA = BC$
State OA is parallel to BC

Q10:

(i) *EITHER*: State a correct expression for $|z|$ or $|z|^2$, e.g. $(1+\cos 2\theta)^2 + (\sin 2\theta)^2$
Use double angle formulae throughout or Pythagoras
Obtain given answer $2\cos\theta$ correctly
State a correct expression for tangent of argument, e.g. $(\sin 2\theta/(1+\cos 2\theta)$
Use double angle formulae to express it in terms of $\cos\theta$ and $\sin\theta$
Obtain $\tan\theta$ and state that the argument is θ
OR: Use double angle formulae to express z in terms of $\cos\theta$ and $\sin\theta$
Obtain a correct expression, e.g. $1+\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$
Convert the expression to polar form
Obtain $2\cos\theta(\cos\theta + i\sin\theta)$
State that the modulus is $2\cos\theta$
State that the argument is θ

(ii) Substitute for z and multiply numerator and denominator by the conjugate of z, or equivalent
Obtain correct real denominator in any form
Identify and obtain real part equal to $\frac{1}{2}$

Q11:

(i) Obtain modulus $\sqrt{8}$
Obtain argument $\frac{1}{4}\pi$ or 45°

(ii) Show 1, i and u in relatively correct positions on an Argand diagram
Show the perpendicular bisector of the line joining 1 and i
Show a circle with centre u and radius 1
Shade the correct region

(iii) State or imply relevance of the appropriate tangent from O to the circle
Carry out complete strategy for finding $|z|$ for the critical point
Obtain answer $\sqrt{7}$

Q12:

(a) *EITHER*: Substitute $1 + i\sqrt{3}$, attempt complete expansions of the x^3 and x^2 terms
Use $i^2 = -1$ correctly at least once
Complete the verification correctly
State that the other root is $1 - i\sqrt{3}$
OR1: State that the other root is $1 - i\sqrt{3}$
State quadratic factor $x^2 - 2x + 4$
Divide cubic by 3-term quadratic reaching partial quotient $2x + k$
Complete the division obtaining zero remainder
OR2: State factorisation $(2x + 3)(x^2 - 2x + 4)$, or equivalent
Make reasonable solution attempt at a 3-term quadratic and use $i^2 = -1$
Obtain the root $1 + i\sqrt{3}$
State that the other root is $1 - i\sqrt{3}$

(b) Show point representing $1 + i\sqrt{3}$ in relatively correct position on an Argand diagram
Show circle with centre at $1 + i\sqrt{3}$ and radius 1
Show line for $\arg z = \frac{1}{3}\pi$ making $\frac{1}{3}\pi$ with the real axis
Show line from origin passing through centre of circle, or the diameter which would contain the origin if produced
Shade the relevant region

Q13:

(i) State modulus is 2
State argument is $\frac{1}{6}\pi$, or 30°, or 0.524 radians

(ii) **(a)** State answer $3\sqrt{3} + i$

(b) *EITHER*: Multiply numerator and denominator by $\sqrt{3} - i$, or equivalent
Simplify denominator to 4 or numerator to $2\sqrt{3} + 2i$
Obtain final answer $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, or equivalent

(iii) Plot A and B in relatively correct positions
EITHER: Use fact that angle $AOB = \arg(iz^*) - \arg z$
Obtain the given answer

Q14:

(i) Attempt multiplication and use $i^2 = -1$
Obtain $3 + 4i$
Obtain 5 for modulus

(ii) Draw complete circle with centre corresponding to their w^2 …
… and radius corresponding to their $|w^2|$
Shade the correct region

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
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Q15:

(i) Either: Multiply numerator and denominator by $(1-2i)$, or equivalent
Obtain $-3i$
State modulus is 3
Refer to u being on negative imaginary axis or equivalent and confirm argument as $-\frac{1}{2}\pi$

Or: Using correct processes, divide moduli of numerator and denominator
Obtain 3
Subtract argument of denominator from argument of numerator
Obtain $-\tan^{-1}\frac{1}{2}-\tan^{-1}2$ or $-0.464-1.107$ and hence $-\frac{1}{2}\pi$ or -1.57

(ii) Show correct half-line from u at angle $\frac{1}{4}\pi$ to real direction
Use correct trigonometry to find required value
Obtain $\frac{3}{2}\sqrt{2}$ or equivalent

(iii) Show, or imply, locus is a circle with centre $(1+i)u$ and radius 1
Use correct method to find distance from origin to furthest point of circle
Obtain $3\sqrt{2}+1$ or equivalent

Q16:

(a) (i) *EITHER*: Multiply numerator and denominator by $a-2i$, or equivalent
Obtain final answer $\frac{5a}{a^2+4}-\frac{10i}{a^2+4}$, or equivalent

(ii) Either state $\arg(u)=-\frac{3}{4}\pi$, or express u^* in terms of a (f.t. on u)
Use correct method to form an equation in a, e.g. $5a=-10$
Obtain $a=-2$ correctly

(b) Show a point representing $2+2i$ in relatively correct position in an Argand diagram
Show the circle with centre at the origin and radius 2
Show the perpendicular bisector of the line segment from the origin to the point representing $2+2i$
Shade the correct region
[SR: Give the first B1 and the B1√ for obtaining $y=2-x$, or equivalent, and sketching the attempt.]

Q17:

(i) Use the quadratic formula, completing the square, or the substitution $z=x+iy$ to find a root and use $i^2=-1$
Obtain final answers $-\sqrt{3}\pm i$, or equivalent

(ii) State that the modulus of both roots is 2
State that the argument of $-\sqrt{3}+i$ is 150° or $\frac{5}{6}\pi$ (2.62) radians
State that the argument of $-\sqrt{3}-i$ is −150° (or 210°) or $-\frac{5}{6}\pi$ (−2.62) radians or $\frac{7}{6}\pi$ (3.67) radians

(iii) Carry out an attempt to find the sixth power of a root
Verify that one of the roots satisfies $z^6=-64$
Verify that the other root satisfies the equation

Q18:

(a) *EITHER*: Square $x+iy$ and equate real and imaginary parts to 1 and $-2\sqrt{6}$ respectively
Obtain $x^2-y^2=1$ and $2xy=-2\sqrt{6}$
Eliminate one variable and find an equation in the other M1
Obtain $x^4-x^2-6=0$ or $y^4+y^2-6=0$, or 3-term equivalent
Obtain answers $\pm(\sqrt{3}-i\sqrt{2})$

(b) Show point representing 3i on a sketch of an Argand diagram
Show a circle with centre at the point representing 3i and radius 2
Shade the interior of the circle
Carry out a complete method for finding the greatest value of arg z
Obtain answer 131.8° or 2.30 (or 2.3) radians
[The f.t. is on solutions where the centre is at the point representing −3i.]

Q19:

(i) Use correct method for finding modulus of their w^2 or w^3 or both
Obtain $|w^2|=2$ and $|w^3|=2\sqrt{2}$ or equivalent
Use correct method for finding argument of their w^2 or w^3 or both
Obtain $\arg(w^2)=-\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ and $\arg(w^3)=\frac{1}{4}\pi$

(ii) Obtain centre $-\frac{1}{2}-\frac{1}{2}i$ (their w^2)
Calculate the diameter or radius using $|w-w^2|$ w21 or right-angled triangle or cosine rule or equivalent
Obtain radius $\frac{1}{2}\sqrt{10}$ or equivalent
Obtain $|z+\frac{1}{2}+\frac{1}{2}i|=\frac{1}{2}\sqrt{10}$ or equivalent

Q20:

(i) Either Expand $(1+2i)^2$ to obtain $-3+4i$ or unsimplified equivalent
Multiply numerator and denominator by $2-i$
Obtain correct numerator $-2+11i$ or correct denominator 5
Obtain $-\frac{2}{5}+\frac{11}{5}i$ or equivalent

Or Expand $(1+2i)^2$ to obtain $-3+4i$ or unsimplified equivalent
Obtain two equations in x and y and solve for x or y
Obtain final answer $x=-\frac{2}{5}$
Obtain final answer $y=\frac{11}{5}$

(ii) Draw a circle
Show centre at relatively correct position, following their u
Draw circle passing through the origin

Q21:

(i) *EITHER*: Multiply numerator and denominator by $1+3i$, or equivalent
Simplify numerator to $-5+5i$, or denominator to 10, or equivalent
Obtain final answer $-\frac{1}{2}+\frac{1}{2}i$, or equivalent

(ii) Show B and C in relatively correct positions in an Argand diagram
Show u in a relatively correct position

(iii) Substitute exact arguments in the LHS $\arg(1+2i)-\arg(1-3i)=\arg u$, or equivalent
Obtain and use $\arg u=\frac{3}{4}\pi$
Obtain the given result correctly

Q22:

(a) EITHER: Eliminate u or w and obtain an equation in w or in u
Obtain a quadratic in u or w, e.g. $u^2-4iu-5=0$ or $w^2+4iw-5=0$
Solve a 3-term quadratic for u or for w

(b) (i) Show point representing $2-2i$ in relatively correct position
Show a circle with centre $2-2i$ and radius 2
Show line for $\arg z=-\frac{1}{4}\pi$
Show line for $\text{Re}\,z=1$
Shade the relevant region

(ii) State answer $2+\sqrt{2}$, or equivalent (accept 3.41)

Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q23:

(i) *EITHER* Substitute $x = 1 + \sqrt{2}$ i and attempt the expansions of the x^2 and x^4 terms
Use $i^2 = -1$ correctly at least once
Complete the verification
State second root $1 - \sqrt{2}$ i

(ii) Carry out a complete method for finding a quadratic factor with zeros $1 \pm \sqrt{2}$ i — M1*
Obtain $x^2 - 2x + 3$, or equivalent — A1
Attempt division of p(x) by $x^2 - 2x + 3$ reaching a partial quotient $x^2 + kx$, or equivalent — M1 (dep*)
Obtain quadratic factor $x^2 - 2x + 2$ — A1
Find the zeros of the second quadratic factor, using $i^2 = -1$ — M1 (dep*)
Obtain roots $-1 + i$ and $-1 - i$ — A1 [6]
[The second M1 is earned if inspection reaches an unknown factor $x^2 + Bx + C$ and an equation in B and/or C, or an unknown factor $Ax^2 + Bx + (6/3)$ and an equation in A and/or B]
[If part **(i)** is attempted by the *OR 1* method, then an attempt at part **(ii)** which uses or quotes relevant working or results obtained in part **(i)** should be marked using the scheme for part **(ii)**]

Q24:

(a) Expand and simplify as far as $\mathrm{i}w^2 = -8\mathrm{i}$ or equivalent
Obtain first answer $\mathrm{i}\sqrt{8}$, or equivalent
Obtain second answer $-\mathrm{i}\sqrt{8}$, or equivalent and no others

(b) **(i)** Draw circle with centre in first quadrant
Draw correct circle with interior shaded or indicated

(ii) Identify ends of diameter corresponding to line through origin and centre
Obtain $p = 3.66$ and $q = 7.66$
Show tangents from origin to circle
Evaluate $\sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$
Obtain $\alpha = \frac{1}{4}\pi - \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 0.424
Obtain $\beta = \frac{1}{4}\pi + \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 1.15