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Trigonometry P3

Q1

(i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [4]

Q2

Solve the equation

$$\tan x \tan 2x = 1,$$

giving all solutions in the interval $0° < x < 180°$. [4]

Q3

Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

Q4

(i) Show that the equation

$$\tan(45° + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(45° + x) - \tan x = 2,$$

giving all solutions in the interval $0° \leqslant x \leqslant 180°$. [4]

Q5

(i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form

$$\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0.$$ [4]

(ii) Hence, or otherwise, solve the equation

$$\tan(30° + \theta) = 2\tan(60° - \theta),$$

for $0° \leqslant \theta \leqslant 180°$. [3]

Q6

(i) Express $5\sin x + 12\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

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Q7

(i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot \theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec} 2\theta + \cot 2\theta = 2$, for $0° \leqslant \theta \leqslant 360°$. [2]

Q8

Prove the identity $\cos 4\theta - 4 \cos 2\theta + 3 \equiv 8 \sin^4 \theta$. [4]

Q9

The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan \alpha = 2 \tan \beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

Q10

It is given that $\cos a = \frac{3}{5}$, where $0° < a < 90°$. Showing your working and without using a calculator to evaluate a,

(i) find the exact value of $\sin(a - 30°)$, [3]

(ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

Q11

Solve the equation

$$\sin \theta = 2 \cos 2\theta + 1,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [6]

Q12

Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

$$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

Q13

Solve the equation

$$\tan(45° - x) = 2 \tan x,$$

giving all solutions in the interval $0° < x < 180°$. [5]

Q14

Prove the identity $\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta$. [4]

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Q15

Solve the equation

$$\cos(\theta + 60°) = 2\sin\theta,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [5]

Q16

(i) Express $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle θ which satisfies the equation

    (a) $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta = -4,$ [2]

    (b) $(\sqrt{6})\cos\frac{1}{2}\theta + (\sqrt{10})\sin\frac{1}{2}\theta = 3.$ [4]

Q17

(i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

    (a)   solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, [3]

    (b)   find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \cos^4\theta \, d\theta.$ [3]

Q18

Solve the equation

$$\cos\theta + 4\cos 2\theta = 3,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [5]

Q19

(i) Show that the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = k$$
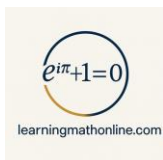
can be written in the form

$$(2\sqrt{3})(1 + \tan^2\theta) = k(1 - 3\tan^2\theta).$$ [4]

(ii) Hence solve the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = 3\sqrt{3},$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [3]

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Q20

(i) Express $\cos x + 3 \sin x$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3 \sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

Q21

(i) Express $8 \cos \theta + 15 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $8 \cos \theta + 15 \sin \theta = 12$, giving all solutions in the interval $0° < \theta < 360°$. [4]

Q22

Solve the equation

$$\operatorname{cosec} 2\theta = \sec \theta + \cot \theta,$$

giving all solutions in the interval $0° < \theta < 360°$. [6]

Q23

It is given that $\tan 3x = k \tan x$, where k is a constant and $\tan x \ne 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1) \tan^2 x = k - 3.$$ [4]

(ii) Hence solve the equation $\tan 3x = k \tan x$ when $k = 4$, giving all solutions in the interval $0° < x < 180°$. [3]

(iii) Show that the equation $\tan 3x = k \tan x$ has no root in the interval $0° < x < 180°$ when $k = 2$. [1]

Q24

Solve the equation

$$\sin(\theta + 45°) = 2 \cos(\theta - 30°),$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Q25

(i) Express $24 \sin \theta - 7 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence find the smallest positive value of θ satisfying the equation

$$24 \sin \theta - 7 \cos \theta = 17.$$ [2]

Answers:

Q1:

(i) State answer $R = 25$
Use trig formula to find α
Obtain $\alpha = 73.74°$

(ii) Carry out evaluation of $\cos^{-1}(15/25)$ $(= 53.1301...°)$
Obtain answer $126.9°$
Carry out correct method for second answer
Obtain answer $20.6°$ and no others in the range
[Ignore answers outside the given range.]

Q2

EITHER: Use $\tan 2A$ formula and obtain a horizontal equation in $\tan x$
Simplify the equation to the form $3\tan^2 x = 1$, or equivalent
Obtain answer $30°$
Obtain second answer $150°$ and no others in the range

Q3:

State answer $R = 2$
Use trig formula to find α
Obtain $\alpha = \frac{1}{3}\pi$, or $60°$

[For the M1 condone a sign error in the expansion of $\cos(\theta - \alpha)$, but the subsequent trigonometric work must be correct.]

[SR: The answer $\alpha = \tan^{-1}(\sqrt{3})$ earns M1 only.]

Q4

(i) Use correct $\tan(A + B)$ formula to obtain an equation in $\tan x$
Use $\tan 45° = 1$
Obtain the given answer

(ii) Make reasonable attempt to solve the given quadratic for one value of $\tan x$
Obtain $\tan x = -1 \pm \sqrt{2}$, or equivalent in the form $(a \pm \sqrt{b})/c$ (accept 0.4, –2.4)
Obtain answer $x = 22.5°$
Obtain second answer $x = 112.5$ and no others in the range
[Ignore answers outside the range.]
[Treat answers in radians as a MR and deduct one mark from the marks for the angles.]

Q5:

(i) Use $\tan(A \pm B)$ formula correctly at least once to obtain an equation in $\tan \theta$
Obtain a correct horizontal equation in any form
Use correct exact values of $\tan 30°$ and $\tan 60°$ throughout
Obtain the given equation correctly

(ii) Make reasonable attempt to solve the given quadratic in $\tan \theta$
Obtain answer $\theta = 24.7°$
Obtain answer $\theta = 95.3°$ and no others in the given range
[Ignore answers outside the given range.]
[Treat answers in radians as MR and deduct one mark from the marks for the angles.]

Q6

(i) State or imply at any stage answer $R = 13$
Use trig formula to find α
Obtain $\alpha = 67.38°$ with no errors seen
[Do not allow radians in this part. If the only trig error is a sign error in $\sin(x + \alpha)$ give M1A0.]

(ii) Evaluate $\sin^{-1}\left(\frac{11}{13}\right)$ correctly to at least 1 d.p $(57.79577...°)$
Carry out an appropriate method to find a value of 2θ in $0° < 2\theta < 360°$
Obtain an answer for θ in the given range, e.g. $\theta = 27.4°$
Use an appropriate method to find another value of 2θ in the above range
Obtain second angle, e.g. $\theta = 175.2°$ and no others in the given range

Q7:

(i) Use $\cot A = 1/\tan A$ or $\cos A/\sin A$ and/or $\csc A = 1/\sin A$ on at least two terms
Use a correct double angle formula or the $\sin(A - B)$ formula at least once
Obtain given result

(ii) Solve $\cot \theta = 2$ for θ and obtain answer $26.6°$
Obtain answer $206.6°$ and no others in the given range

Q8

Use double angle formulae correctly to express LHS in terms of trig functions of 2θ
Use trig formulae correctly to express LHS in terms of $\sin \theta$, converting at least two terms
Obtain expression in any correct form in terms of $\sin \theta$

Q9:

Use $\tan(A \pm B)$ formula and obtain an equation in $\tan \alpha$ and $\tan \beta$
Substitute throughout for $\tan \alpha$ or for $\tan \beta$
Obtain $2\tan^2 \beta + \tan \beta - 1 = 0$ or $\tan^2 \alpha + \tan \alpha - 2 = 0$, or equivalent
Solve a 3-term quadratic and find an angle
Obtain answer $\alpha = 45°$, $\beta = 26.6°$
Obtain answer $\alpha = 116.6°$, $\beta = 135°$

Q10

(i) State or imply $\sin a = 4/5$
Use $\sin(A - B)$ formula and substitute for $\cos a$ and $\sin a$
Obtain answer $\frac{1}{10}(4\sqrt{3} - 3)$, or exact eqivalent

(ii) Use $\tan 2A$ formula and substitute for $\tan a$, or use $\sin 2A$ and $\cos 2A$ formulae, substitute $\sin a$ and $\cos a$, and divide
Obtain $\tan 2a = -\frac{24}{7}$, or equivalent
Use $\tan(A + B)$ formula with $A = 2a$, $B = a$ and substitute for $\tan 2a$ and $\tan a$
Obtain $\tan 3a = -\frac{44}{117}$

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Q11:

Use correct cos $2A$ formula and obtain an equation in sin θ

Obtain $4\sin^2\theta + \sin\theta - 3 = 0$, or equivalent

Make reasonable attempt to solve a 3-term quadratic in sin θ

Obtain answer $48.6°$

Obtain answer $131.4°$ and no others in the given range

Obtain answer $270°$ and no others in the given range

Q12:

State correct expansion of $\cos(3x - x)$ or $\cos(3x + x)$

Substitute expansions in $\frac{1}{2}(\cos 2x - \cos 4x)$, or equivalent

Simplify and obtain the given identity correctly

Q13:

Attempt to use $\tan(A \pm B)$ formula and obtain an equation in $\tan x$

Obtain 3-term quadratic $2\tan^2 x + 3\tan x - 1 = 0$, or equivalent

Solve a 3-term quadratic and find a numerical value of x

Obtain answer $15.7°$

Obtain answer $119.3°$ and no others in the given interval

Q14

Use correct $\cos(A + B)$ formula to express $\cos 3\theta$ in terms of trig functions of 2θ and θ

Use correct trig formulae and Pythagoras to express $\cos 3\theta$ in terms of $\cos\theta$

Obtain a correct expression in terms of $\cos\theta$ in any form

Obtain the given identity correctly

Q15:

Obtain answer $\theta = 9.9°$

Obtain $\theta = 189.9°$, and no others in the given interval

[Ignore answers outside the given interval. Treat answers in radians as a misread $(0.173, 3.31)$.]

[The other solution methods are *via* $\cos\theta = \pm(4 + \sqrt{3})/\sqrt{\left(1+\left(4+\sqrt{3}\right)^2\right)}$ or

$\sin\theta = \pm 1/\sqrt{\left(1+\left(4+\sqrt{3}\right)^2\right)}$.]

Q16

(i) Obtain or imply $R = 4$

Use appropriate trigonometry to find α

Obtain $\alpha = 52.24$ or better from correct work

(ii) (a) State or imply $\theta - \alpha = \cos^{-1}(-4 \div R)$
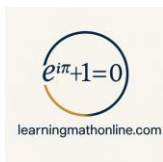
Obtain 232.2 or better

(b) Attempt at least one value using $\cos^{-1}(3 \div R)$

Obtain one correct value e.g. $\pm 41.41°$

Use $\frac{1}{2}\theta - \alpha = \cos^{-1}\left(\frac{3}{R}\right)$ to find θ

Obtain 21.7

Q17:

(i) Express $\cos 4\theta$ as $2\cos^2 2\theta - 1$ or $\cos^2 2\theta - \sin^2 2\theta$ or $1 - 2\sin^2 2\theta$

Express $\cos 4\theta$ in terms of $\cos\theta$

Obtain $8\cos^4\theta - 8\cos^2\theta + 1$

Use $\cos 2\theta = 2\cos^2\theta - 1$ to obtain given answer $8\cos^4\theta - 3$ **AG**

(ii) (a) State or imply $\cos^4\theta = \frac{1}{2}$

Obtain 0.572

Obtain -0.572

(b) Integrate and obtain form $k_1\theta + k_2\sin 4\theta + k_3\sin 2\theta$

Obtain $\frac{3}{8}\theta + \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta$

Obtain $\frac{3}{32}\pi + \frac{1}{4}$ following completely correct work

Q18

Obtain answer $29.0°$

Obtain answer $180°$ ε

Q19:

(ii) Set $k = 3\sqrt{3}$ and obtain $\tan^2\theta = \frac{1}{11}$

Obtain answer $16.8°$

Obtain answer $163.2°$

Q20

(i) State or imply $R = \sqrt{10}$

Use trig formulae to find α

Obtain $\alpha = 71.57°$ with no errors seen

[Do not allow radians in this part. If the only trig error is a sign error in $\cos(x - \alpha)$ give M1A0]

(ii) Evaluate $\cos^{-1}(2/\sqrt{10})$ correctly to at least 1 d.p. $(50.7684...°)$ (Allow $50.7°$ here)

Carry out an appropriate method to find a value of 2θ in $0° < 2\theta < 180°$

Obtain an answer for θ in the given range, e.g. $\theta = 61.2°$

Use an appropriate method to find another value of 2θ in the above range

Obtain second angle, e.g. $\theta = 10.4°$, and no others in the given range

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Q21:

(i) State or imply $R = 17$

Use correct trigonometric formula to find α

Obtain 61.93° with no errors seen

(ii) Evaluate $\cos^{-1}\frac{12}{R}$ ($= 45.099$)

Obtain answer 107.0°

Carry out correct method for second answer

Obtain answer 16.8° and no others between 0° and 360°

Q22

Use trig formulae to express equation in terms of $\cos\theta$ and $\sin\theta$
Use Pythagoras to obtain an equation in $\sin\theta$
Obtain 3-term quadratic $2\sin^2\theta - 2\sin\theta - 1 = 0$, or equivalent
Solve a 3-term quadratic and obtain a value of θ
Obtain answer, e.g. 201.5°
Obtain second answer, e.g. 338.5°, and no others in the given interval

Q23:

(ii) Substitute $k = 4$ in the given expression and solve for x
Obtain answer, e.g. $x = 16.8°$
Obtain second answer, e.g. $x = 163.2°$, and no others in the given interval
[Ignore answers outside the given interval. Treat answers in radians as a misread and deduct A1 from the marks for the angles.]

(iii) Substitute $k = 2$, show $\tan^2 x < 0$ and justify given statement correctly

Q24 (105.9)

Q25:

(i) State or imply $R = 25$
Use correct trigonometric formula to find α
Obtain 16.26° **with no errors seen**

(ii) Evaluate of $\sin^{-1}\frac{17}{R}$ ($= 42.84\ldots°$)
Obtain answer 59.1°