Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Differentiation – Implicit & Parametric P3

Q1

The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \quad y = 1 - \cos 2\theta.$$

Show that $\frac{dy}{dx} = \tan\theta$. [5]

Q2

The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Q3

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

Q4

The equation of a curve is $y = x \sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

Q5

The curve with equation $y = e^{-x} \sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Q6

The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [8]

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q7

The curve $y = \dfrac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the x-coordinate of this point. [5]

Q8

The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \qquad y = a(1 - \cos 2\theta).$$

Show that $\dfrac{dy}{dx} = \cot\theta$. [5]

Q9

The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\dfrac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin t + y\cos t = a\sin t\cos t.$$ [3]

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

Q10

A curve has equation $y = e^{-3x}\tan x$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Give your answers correct to 3 decimal places. [6]

Q11

The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the equation of the tangent to the curve at the point $(2, 1)$, giving your answer in the form $ax + by + c = 0$. [2]

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q12

The equation of a curve is

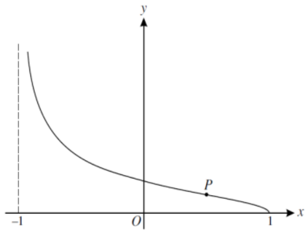
$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

Q13

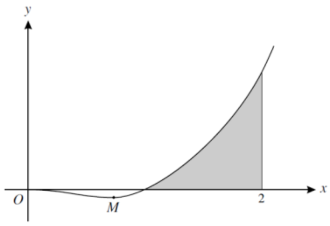
The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.



(i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. [5]

(ii) The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. [4]

Q14



The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

Find the exact coordinates of M. [5]

Q15

The parametric equations of a curve are

$$x = \frac{t}{2t+3}, \qquad y = e^{-2t}.$$

Find the gradient of the curve at the point for which $t = 0$. [5]

Q16

Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = \ln(1 + \sin 2x)$, [2]

(ii) $y = \frac{\tan x}{x}$. [2]

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Facebook/Learning Math Online

Q17

The parametric equations of a curve are

$$x = \ln(\tan t), \quad y = \sin^2 t,$$

where $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [4]

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. [3]

Q18

The curve $y = \frac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. [4]

Q19

The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

Q20

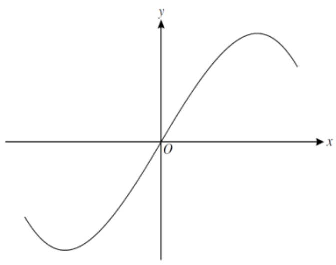
The equation of a curve is $y = \frac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]

Q21

The diagram shows the curve with parametric equations

$$x = \sin t + \cos t, \quad y = \sin^3 t + \cos^3 t,$$

for $\frac{1}{4}\pi < t < \frac{5}{4}\pi$.



(i) Show that $\frac{dy}{dx} = -3\sin t \cos t$. [3]

(ii) Find the gradient of the curve at the origin. [2]

(iii) Find the values of t for which the gradient of the curve is 1, giving your answers correct to 2 significant figures. [4]

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Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q22

The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

(i) Find the gradient of the curve at the point $(2, -3)$. [4]

(ii) Show that there are no points on the curve at which the gradient is 1. [3]

Q23

The equation of a curve is $y = 3\sin x + 4\cos^3 x$.

(i) Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. [6]

(ii) Determine the nature of the stationary point in this interval for which x is least. [2]

Q24

The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \quad y = \cos 2\theta + 2\sin\theta.$$

Show that $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{1 + 2\sin\theta}$. [5]

Q25

The curve with equation $y = \dfrac{e^{2x}}{x^3}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Q26

The equation of a curve is $\ln(xy) - y^3 = 1$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y}{x(3y^3 - 1)}$. [4]

(ii) Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. [4]

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Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q27

The parametric equations of a curve are

$$x = \frac{4t}{2t+3}, \quad y = 2\ln(2t+3).$$

(i) Express $\frac{dy}{dx}$ in terms of t, simplifying your answer. [4]

(ii) Find the gradient of the curve at the point for which $x = 1$. [2]

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
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Facebook/Learning Math Online

Answers:

Q1:

State that $\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ or $\frac{dy}{d\theta} = 2\sin 2\theta$

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

Obtain answer in any correct form, e.g. $\frac{2\sin 2\theta}{2 + 2\cos 2\theta}$

Make relevant use of sin 2A and cos 2A formulae

Obtain given answer correctly

Q2:

(i) State derivative is $6e^x - 3e^{3x}$

EITHER: Equate derivative to zero and simplify to an equation of the form $e^{2x} = a$

Carry out method for calculating x, where $a > 0$

Obtain answer $x = \frac{1}{2}\ln 2$, or equivalent (0.347, or 0.346, or 0.35)

(ii) Carry out a method for determining the nature of a stationary point

Show that the point is a maximum with no errors seen

Q3:

(i) State $2(3y^2)\frac{dy}{dx}$ as derivative of $2y^3$, or equivalent

State $3x\frac{dy}{dx} + 3y$ as derivative of $3xy$, or equivalent

Solve for $\frac{dy}{dx}$

Obtain given answer correctly

[The M1 is dependent on at least one of the B marks being obtained.]

(ii) State or imply that the coordinates satisfy $y - x^2 = 0$

Obtain an equation in x (or in y)

Solve and obtain $x = 1$ only (or $y = 1$ only)

Substitute x- (or y-)value in $y - x^2 = 0$ or in the equation of the curve

Obtain $y = 1$ only (or $x = 1$ only)

Q4:

Use product rule

Obtain derivative in any correct form

Form equation of tangent at $x = \frac{1}{4}\pi$ correctly

Simplify answer to $y = x$, or $y - x = 0$

[SR: The misread $y = x\sin x$ can only earn M1M1.]

Q5:

(i) Use correct product or quotient rule

Obtain derivative in any correct form

Equate derivative to zero and solve for x

Obtain answer $x = \frac{1}{4}\pi$ or 0.785 with no errors seen

(ii) Use an appropriate method for determining the nature of a stationary point

Show the point is a maximum point with no errors seen

[SR: for the answer 45° deduct final A1 in part (i), and deduct A1 in part (ii) if this value in degrees is used in the exponential.]

Q6:

State $x^2\frac{dy}{dx} + 2xy$, or equivalent, as derivative of x^2y

State $y^2 + 2xy\frac{dy}{dx}$, or equivalent, as derivative of xy^2

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Facebook/Learning Math Online

Q7:

Use correct quotient or product rule

Obtain correctly the derivative in any form, e.g. $\dfrac{e^x \cos x + e^x \sin x}{\cos^2 x}$

Equate derivative to zero and reach $\tan x = k$

Solve for x

Obtain $x = -\frac{1}{4}\pi$ (or -0.785) only (accept x in $[-0.79, -0.78]$ but not in degrees)

[The last three marks are independent. Fallacious log work forfeits the M1*. For the M1(dep*) the solution can lie outside the given range and be in degrees, but the mark is not available if $k = 0$. The final A1 is only given for an entirely correct answer to the whole question.]

Q8:

State or imply $\dfrac{dx}{d\theta} = a(2 - 2\cos 2\theta)$ or $\dfrac{dy}{d\theta} = 2a\sin 2\theta$

Use $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta}$

Obtain $\dfrac{dy}{dx} = \dfrac{\sin 2\theta}{(1 - \cos 2\theta)}$, or equivalent

Make use of correct $\sin 2A$ and $\cos 2A$ formulae

Obtain the given result following sufficient working

[SR: An attempt which assumes a is the parameter and θ a constant can only earn the two M marks. One that assumes θ is the parameter and a is a function of θ can earn B1M1A0M1A0.]

[SR: For an attempt that gives a a value, e.g. 1, or ignores a, give B0 but allow the remaining marks.]

Q9:

(i) State $\dfrac{dx}{dt} = -3a\cos^2 t \sin t$ or $\dfrac{dy}{dt} = 3a\sin^2 t \cos t$, or equivalent

Use $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

(ii) Form the equation of the tangent

Obtain the equation in any correct form

Obtain the given answer

(iii) State the x-coordinate of X or the y-coordinate of Y in any correct form

Obtain the given answer with no errors seen

Q10:

Use product or quotient rule

Obtain derivative in any correct form

Equate derivative to zero and obtain an equation of the form $a\sin 2x = b$, or a quadratic in $\tan x$, $\sin^2 x$, or $\cos^2 x$

Carry out correct method for finding one angle

Obtain answer, e.g. 0.365

Obtain second answer 1.206 and no others in the range (allow 1.21)

[Ignore answers outside the given range.]

[Treat answers in degrees, 20.9° and 69.1°, as a misread.]

Q11:

(i) State $2xy + x^2 \dfrac{dy}{dx}$ as derivative of $x^2 y$

State $3y^2 \dfrac{dy}{dx}$ as derivative of y^3

Equate derivative of LHS to zero and solve for $\dfrac{dy}{dx}$

Obtain answer $\dfrac{3x^2 - 2xy}{x^2 + 3y^2}$, or equivalent

(ii) Find gradient of tangent at (2, 1) and form equation of tangent

Obtain answer $8x - 7y - 9 = 0$, or equivalent

Q12:

(i) State or imply $\dfrac{1}{y}\dfrac{dy}{dx}$ as derivative of $\ln y$

State correct derivative of LHS, e.g. $\ln y + \dfrac{x}{y}\dfrac{dy}{dx}$

Differentiate RHS and obtain an expression for $\dfrac{dy}{dx}$

Obtain given answer

(ii) State or imply $x = -\frac{1}{2}$ when $y = 1$

Substitute and obtain gradient of -4

Correctly form equation of tangent

Obtain final answer $y + 4x + 1 = 0$, or equivalent

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q13:

(i) Use quotient or product rule to differentiate $(1-x)/(1+x)$
Obtain correct derivative in any form
Use chain rule to find $\frac{dy}{dx}$
Obtain a correct expression in any form
Obtain the gradient of the normal in the given form correctly

(ii) Use product rule
Obtain correct derivative in any form
Equate derivative to zero and solve for x
Obtain $x = \frac{1}{2}$

Q14:

Use correct product rule
Obtain correct derivative in any form
Equate derivative to zero and find non-zero x
Obtain $x = \exp(-\frac{1}{3})$, or equivalent
Obtain $y = -1/(3e)$, or any ln-free equivalent

Q15:

Use of correct quotient or product rule to differentiate x or t
Obtain correct $\frac{3}{(2t+3)^2}$ or unsimplified equivalent
Obtain $-2e^{-2t}$ for derivative of y
Use $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ or equivalent
Obtain -6

Alternative:

Eliminate parameter and attempt differentiation $\left(y = e^{\frac{-6x}{1-2x}}\right)$
Use correct quotient or product rule
Use chain rule
Obtain $\frac{dy}{dx} = \frac{-6}{(1-2x)^2}e^{\frac{-6x}{1-2x}}$
Obtain -6

Q16:

(i) Obtain $\frac{k\cos 2x}{1+\sin 2x}$ for any non-zero constant k
Obtain $\frac{2\cos 2x}{1+\sin 2x}$

(ii) Use correct quotient or product rule
Obtain $\frac{x\sec^2 x - \tan x}{x^2}$ or equivalent

Q17:

(i) *EITHER*: State $\frac{dx}{dt} = \sec^2 t / \tan t$, or equivalent
State $\frac{dy}{dt} = 2\sin t\cos t$, or equivalent
Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

(ii) State or imply $t = \frac{1}{4}\pi$ when $x = 0$
Form the equation of the tangent at $x = 0$
Obtain correct answer in any horizontal form, e.g. $y = \frac{1}{2}x + \frac{1}{2}$

[SR: If the *OR* method is used in part (i), give B1 for stating or implying $y = \frac{1}{2}$ or $\frac{dy}{dx} = \frac{1}{2}$ when $x = 0$.]

Q18:

Use correct quotient or product rule
Obtain correct derivative in any form, e.g. $-\frac{3\ln x}{x^4} + \frac{1}{x^4}$
Equate derivative to zero and solve for x an equation of the form $\ln x = a$, where $a > 0$
Obtain answer $\exp(\frac{1}{3})$, or 1.40, from correct work

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q19:

Use chain rule

obtain $\frac{dx}{dt} = 6\sin t\cos t$, or equivalent

obtain $\frac{dy}{dt} = -6\cos^2 t\sin t$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain final answer $\frac{dy}{dx} = -\cos t$

Q20:

Use correct quotient or product rule or equivalent

Obtain $\frac{(1+e^{2x}).2e^{2x} - e^{2x}.2e^{2x}}{(1+e^{2x})^2}$ or equivalent

Substitute $x = \ln 3$ into attempt at first derivative and show use of relevant logarithm property at least once in a correct context

Confirm given answer $\frac{9}{50}$ legitimately

Q21:

(i) Differentiate y to obtain $3\sin^2 t\cos t - 3\cos^2 t\sin t$ o.e.

Use $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dt}{dx}$

Obtain given result $-3\sin t\cos t$

(ii) Identify parameter at origin as $t = \frac{3}{4}\pi$

Use $t = \frac{3}{4}\pi$ to obtain $\frac{3}{2}$

(iii) Rewrite equation as equation in one trig variable
e.g. $sin2t = -\frac{2}{3}$, $9\sin^4 x - 9\sin^2 x + 1 = 0$, $\tan^2 x + 3\tan x + 1 = 0$

Find at least one value of t from equation of form $\sin 2t = k$ o.e.

Obtain 1.9

Obtain 2.8 and no others

Q22:

(i) Obtain $2y\frac{dy}{dx}$ as derivative of y^2

Obtain $-4y - 4x\frac{dy}{dx}$ as derivative of $-4xy$

Substitute $x = 2$ and $y = -3$ and find value of $\frac{dy}{dx}$

(dependent on at least one B1 being earned and $\frac{d(45)}{dx} = 0$)

Obtain $\frac{12}{7}$ or equivalent

(ii) Substitute $\frac{dy}{dx} = 1$ in an expression involving $\frac{dy}{dx}$, x and y and obtain $ay = bx$

Obtain $y = x$ or equivalent
Uses $y = x$ in original equation and demonstrate contradiction

Q23:

(i) State derivative in any correct form, e.g. $3\cos x - 12\cos^2 x\sin x$
Equate derivative to zero and solve for $\sin 2x$, or $\sin x$ or $\cos x$

Obtain answer $x = \frac{1}{12}\pi$

Obtain answer $x = \frac{5}{12}\pi$

Obtain answer $x = \frac{1}{2}\pi$ and no others in the given interval

(ii) Carry out a method for determining the nature of the relevant stationary point

Obtain a maximum at $\frac{1}{12}\pi$ correctly

[Treat answers in degrees as a misread and deduct A1 from the marks for the angles.]

Q24:

Obtain $\frac{dx}{d\theta} = 2\cos 2\theta - 1$ or $\frac{dy}{d\theta} = -2\sin 2\theta + 2\cos\theta$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

Obtain $\frac{dy}{dx} = \frac{-2\sin 2\theta + 2\cos\theta}{2\cos 2\theta - 1}$, or equivalent

At any stage use correct double angle formulae throughout
Obtain the given answer following full and correct working

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Fayzan Munawar
Associate *of* Royal College of Science, UK
BSc Mathematics *(Imperial College London)*
tutoring@learningmathonline.com WhatsApp: +1 718 200 2476
Facebook/Learning Math Online

Q25:

(i) Use correct quotient or product rule

Obtain correct derivative in any form, e.g. $\frac{2e^{2x}}{x^3} - \frac{3e^{2x}}{x^4}$

Equate derivative to zero and solve a 2-term equation for non-zero x

Obtain $x = \frac{3}{2}$ correctly

(ii) Carry out a method for determining the nature of a stationary point, e.g. test derivative either side

Show point is a minimum with no errors seen

Q26:

(i) *EITHER*: State or imply $\frac{1}{x} + \frac{1}{y}\frac{dy}{dx}$ as derivative of $\ln xy$, or equivalent

State or imply $3y^2\frac{dy}{dx}$ as derivative of y^3, or equivalent

Equate derivative of LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate denominator to zero and solve for y

Obtain $y = 0.693$ only

Substitute found value in the equation and solve for x

Obtain $x = 5.47$ only

Q27:

(i) Either Use correct quotient rule or equivalent to obtain

$\frac{dx}{dt} = \frac{4(2t+3) - 8t}{(2t+3)^2}$ or equivalent

Obtain $\frac{dy}{dt} = \frac{4}{2t+3}$ or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ or equivalent

Obtain $\frac{1}{3}(2t+3)$ or similarly simplified equivalent

(ii) Obtain $2t = 3$ or $t = \frac{3}{2}$

Substitute in expression for $\frac{dy}{dx}$ and obtain 2

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