

Differentiation

Q1

Differentiate $4x + \frac{6}{x^2}$ with respect to x .

[2]

Q2

The diagram shows a glass window consisting of a rectangle of height h m and width $2r$ m and a semicircle of radius r m. The perimeter of the window is 8 m.

(i) Express h in terms of r .

[2]

(ii) Show that the area of the window, A m², is given by

$$A = 8r - 2r^2 - \frac{1}{2}\pi r^2.$$

[2]

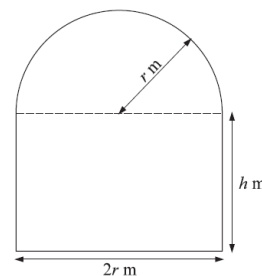
Given that r can vary,

(iii) find the value of r for which A has a stationary value,

[4]

(iv) determine whether this stationary value is a maximum or a minimum.

[2]



Q3

A curve is such that $\frac{dy}{dx} = 3x^2 - 4x + 1$. The curve passes through the point (1, 5).

(i) Find the equation of the curve.

[3]

(ii) Find the set of values of x for which the gradient of the curve is positive.

[3]

Q4

A solid rectangular block has a base which measures $2x$ cm by x cm. The height of the block is y cm and the volume of the block is 72 cm³.

(i) Express y in terms of x and show that the total surface area, A cm², of the block is given by

$$A = 4x^2 + \frac{216}{x}.$$

[3]

Given that x can vary,

(ii) find the value of x for which A has a stationary value,

[3]

(iii) find this stationary value and determine whether it is a maximum or a minimum.

[3]

Q5

A curve has equation $y = x^2 + \frac{2}{x}$.

(i) Write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

[3]

(ii) Find the coordinates of the stationary point on the curve and determine its nature.

[4]

Q6

Find the gradient of the curve $y = \frac{12}{x^2 - 4x}$ at the point where $x = 3$.

[4]

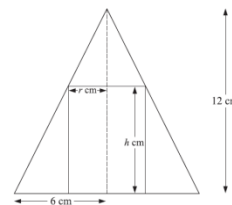
Q7

The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.

- (i) Express h in terms of r and hence show that the volume, $V \text{ cm}^3$, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3. \quad [3]$$

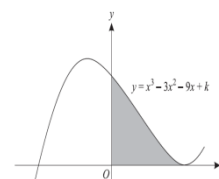
- (ii) Given that r varies, find the stationary value of V . [4]



Q8

The diagram shows the curve $y = x^3 - 3x^2 - 9x + k$, where k is a constant. The curve has a minimum point on the x -axis.

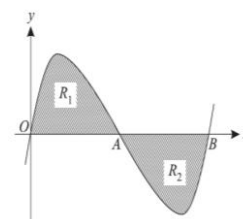
- (i) Find the value of k . [4]
(ii) Find the coordinates of the maximum point of the curve. [1]



Q9

The diagram shows the curve $y = x(x-1)(x-2)$, which crosses the x -axis at the points $O(0, 0)$, $A(1, 0)$ and $B(2, 0)$.

- (i) The tangents to the curve at the points A and B meet at the point C . Find the x -coordinate of C . [5]



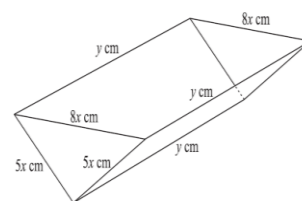
Q10

The diagram shows an open container constructed out of 200 cm^2 of cardboard. The two vertical end pieces are isosceles triangles with sides $5x \text{ cm}$, $5x \text{ cm}$ and $8x \text{ cm}$, and the two side pieces are rectangles of length $y \text{ cm}$ and width $5x \text{ cm}$, as shown. The open top is a horizontal rectangle.

- (i) Show that $y = \frac{200 - 24x^2}{10x}$. [3]
(ii) Show that the volume, $V \text{ cm}^3$, of the container is given by $V = 240x - 28.8x^3$. [2]

Given that x can vary,

- (iii) find the value of x for which V has a stationary value, [3]
(iv) determine whether it is a maximum or a minimum stationary value. [2]



Q11

The equation of a curve is $y = (2x-3)^3 - 6x$.

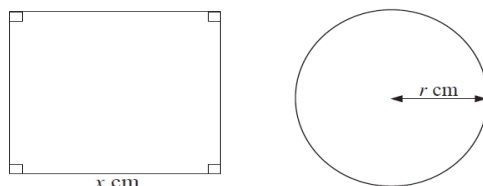
- (i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x . [3]
(ii) Find the x -coordinates of the two stationary points and determine the nature of each stationary point. [5]

Q12

The equation of a curve is $y = 2x + \frac{8}{x^2}$.

- (i) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]
- (ii) Find the coordinates of the stationary point on the curve and determine the nature of the stationary point. [3]

Q13



A wire, 80 cm long, is cut into two pieces. One piece is bent to form a square of side x cm and the other piece is bent to form a circle of radius r cm (see diagram). The total area of the square and the circle is A cm².

- (i) Show that $A = \frac{(\pi + 4)x^2 - 160x + 1600}{\pi}$. [4]
- (ii) Given that x and r can vary, find the value of x for which A has a stationary value. [4]

Q14

The equation of a curve is $y = \frac{12}{x^2 + 3}$.

- (i) Obtain an expression for $\frac{dy}{dx}$. [2]
- (ii) Find the equation of the normal to the curve at the point $P(1, 3)$. [3]
- (iii) A point is moving along the curve in such a way that the x -coordinate is increasing at a constant rate of 0.012 units per second. Find the rate of change of the y -coordinate as the point passes through P . [2]

Q15

A curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 6$ and the point $(9, 2)$ lies on the curve.

- (i) Find the equation of the curve. [4]
- (ii) Find the x -coordinate of the stationary point on the curve and determine the nature of the stationary point. [3]

Q16

A solid rectangular block has a square base of side x cm. The height of the block is h cm and the total surface area of the block is 96 cm^2 .

- (i) Express h in terms of x and show that the volume, $V \text{ cm}^3$, of the block is given by

$$V = 24x - \frac{1}{2}x^3. \quad [3]$$

Given that x can vary,

- (ii) find the stationary value of V , [3]
(iii) determine whether this stationary value is a maximum or a minimum. [2]

Q17

The equation of a curve is $y = \frac{1}{6}(2x - 3)^3 - 4x$.

- (i) Find $\frac{dy}{dx}$. [3]
(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y -axis. [3]
(iii) Find the set of values of x for which $\frac{1}{6}(2x - 3)^3 - 4x$ is an increasing function of x . [3]

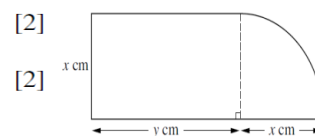
Q18

The diagram shows a metal plate consisting of a rectangle with sides x cm and y cm and a quarter-circle of radius x cm. The perimeter of the plate is 60 cm.

- (i) Express y in terms of x . [2]
(ii) Show that the area of the plate, $A \text{ cm}^2$, is given by $A = 30x - x^2$. [2]

Given that x can vary,

- (iii) find the value of x at which A is stationary, [2]
(iv) find this stationary value of A , and determine whether it is a maximum or a minimum value. [2]



Q19

The length, x metres, of a Green Anaconda snake which is t years old is given approximately by the formula

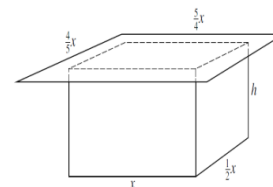
$$x = 0.7\sqrt{(2t - 1)},$$

where $1 \leq t \leq 10$. Using this formula, find

- (i) $\frac{dx}{dt}$, [2]
(ii) the rate of growth of a Green Anaconda snake which is 5 years old. [2]

Q20

The diagram shows an open rectangular tank of height h metres covered with a lid. The base of the tank has sides of length x metres and $\frac{1}{2}x$ metres and the lid is a rectangle with sides of length $\frac{5}{4}x$ metres and $\frac{4}{3}x$ metres. When full the tank holds 4 m^3 of water. The material from which the tank is made is of negligible thickness. The external surface area of the tank together with the area of the top of the lid is $A \text{ m}^2$.



(i) Express h in terms of x and hence show that $A = \frac{3}{2}x^2 + \frac{24}{x}$. [5]

(ii) Given that x can vary, find the value of x for which A is a minimum, showing clearly that A is a minimum and not a maximum. [5]

Q21

A curve has equation $y = \frac{1}{x-3} + x$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [2]

(ii) Find the coordinates of the maximum point A and the minimum point B on the curve. [5]

Q22

The volume of a spherical balloon is increasing at a constant rate of 50 cm^3 per second. Find the rate of increase of the radius when the radius is 10 cm . [Volume of a sphere = $\frac{4}{3}\pi r^3$.] [4]

Q23

The variables x , y and z can take only positive values and are such that

$$z = 3x + 2y \quad \text{and} \quad xy = 600.$$

(i) Show that $z = 3x + \frac{1200}{x}$. [1]

(ii) Find the stationary value of z and determine its nature. [6]

Q24

A curve is such that $\frac{dy}{dx} = \frac{3}{(1+2x)^2}$ and the point $(1, \frac{1}{2})$ lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the set of values of x for which the gradient of the curve is less than $\frac{1}{3}$. [3]

Q25

A curve has equation $y = \frac{4}{3x-4}$ and $P(2, 2)$ is a point on the curve.

(i) Find the equation of the tangent to the curve at P . [4]

(ii) Find the angle that this tangent makes with the x -axis. [2]



Q26

Differentiate $\frac{2x^3 + 5}{x}$ with respect to x . [3]

Q27

A curve is such that $\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 1$ and $P(9, 5)$ is a point on the curve.

- (i) Find the equation of the curve. [4]
- (ii) Find the coordinates of the stationary point on the curve. [3]
- (iii) Find an expression for $\frac{d^2y}{dx^2}$ and determine the nature of the stationary point. [2]
- (iv) The normal to the curve at P makes an angle of $\tan^{-1} k$ with the positive x -axis. Find the value of k . [2]

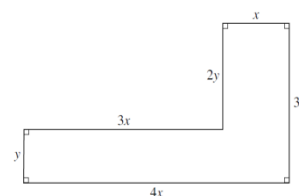
Q28

A curve has equation $y = 3x^3 - 6x^2 + 4x + 2$. Show that the gradient of the curve is never negative. [3]

Q29

The diagram shows the dimensions in metres of an L-shaped garden. The perimeter of the garden is 48 m.

- (i) Find an expression for y in terms of x . [1]
- (ii) Given that the area of the garden is $A \text{ m}^2$, show that $A = 48x - 8x^2$. [2]
- (iii) Given that x can vary, find the maximum area of the garden, showing that this is a maximum value rather than a minimum value. [4]



Q30

A curve $y = f(x)$ has a stationary point at $P(3, -10)$. It is given that $f'(x) = 2x^2 + kx - 12$, where k is a constant.

- (i) Show that $k = -2$ and hence find the x -coordinate of the other stationary point, Q . [4]
- (ii) Find $f''(x)$ and determine the nature of each of the stationary points P and Q . [2]
- (iii) Find $f(x)$. [4]

Q31

A watermelon is assumed to be spherical in shape while it is growing. Its mass, M kg, and radius, r cm, are related by the formula $M = kr^3$, where k is a constant. It is also assumed that the radius is increasing at a constant rate of 0.1 centimetres per day. On a particular day the radius is 10 cm and the mass is 3.2 kg. Find the value of k and the rate at which the mass is increasing on this day. [5]

Answers:

Q1: $4 - 12x^{-3}$

Q2: (i) $h = 4 - r - \frac{1}{2}\pi r$ (ii) $8r - 2r^2 - \frac{1}{2}\pi r^2$ (iii) $\frac{dA}{dr} = 8 - 4r - \pi r$, $r = \frac{8}{4+\pi}$

(iv) $\frac{dA^2}{dr^2} = -4 - \pi$ (negative, so Maximum)

Q3: (i) $y = x^3 - 2x^2 + x$ (+c)
(1,5) used to give c=5

(ii) $3x^2 - 4x + 1 > 0$
→ end values of 1 and $\frac{1}{3}$
→ $x < \frac{1}{3}$ and $x > 1$

Q7: (i) Similar triangles or trig (tan = opp/hyp)
 $\frac{6}{12} = \frac{r}{12-h}$ → $h = 12 - 2r$
→ $V = \pi r^2 h$ → $V = 12\pi r^2 - 2\pi r^3$
(ii) $dV/dr = 24\pi r - 6\pi r^2$
= 0 when $r = 4$ → $V = 64\pi$ (or 201)

Q4: (i) $y = 72 \div (2x^2)$ or $36 \div x^2$
 $A = 4x^2 + 6xy$
→ $A = 4x^2 + 216 \div x$
(ii) $dA/dx = 8x - 216 \div x^2$
= 0 when $8x^3 = 216$
→ $x = 3$
(iii) Stationary value = 108 cm²
 $d^2A/dx^2 = 8 + 432 \div x^3$
→ Positive when $x=3$ Minimum.

Q5: (i) $dy/dx = 2x - 2/x^2$
 $d^2y/dx^2 = 2 + 4/x^3$
(ii) $dy/dx = 0$ $2x - 2/x^2 = 0$
→ $x^3 = 1$ → $x = 1$, $y = 3$
If $x = 1$, $d^2y/dx^2 > 0$, Minimum

Q6: $y = \frac{12}{x^2 - 4x}$
 $\frac{dy}{dx} = -12(x^2 - 4x)^{-2} \times (2x - 4)$
If $x = 3$, $\frac{dy}{dx} = -\frac{8}{3}$

Q8: (i) $y = x^3 - 3x^2 - 9x + k$
 $\frac{dy}{dx} = 3x^2 - 6x - 9$
= 0 when $x = 3$ or $x = -1$
→ $x = 3$, $y = 0$ → $k = 27$
(ii) $x = -1$ → $y = 32$

Q9: (i) $y = x^3 - 3x^2 + 2x$
 $\frac{dy}{dx} = 3x^2 - 6x + 2$
At A (1,0), $m = -1$ → $y = -(x-1)$
At B (2,0), $m = 2$ → $y = 2(x-2)$
Sim equations → $x = \frac{2}{3}$

Q10: (i) Height = $3x$
 $10xy + \frac{1}{2} \cdot 8x \cdot 3x \cdot 2 = 200$
→ $y = \frac{200 - 24x^2}{10x}$
(ii) $V = \frac{1}{2} \cdot 8x \cdot 3xy = 24(3x - 2.8x^2)$
(iii) $\frac{dV}{dx} = 240 - 86.4x^2$
= 0 when $x = 1\frac{2}{3}$
(iv) $\frac{d^2V}{dx^2} = -172.8x$
→ -ve → Maximum

Q11: 8. $y = (2x-3)^3 - 6x$
(i) $\frac{dy}{dx} = 3 \times (2x-3)^2 \times 2 - 6$
 $\frac{d^2y}{dx^2} = 12 \times (2x-3) \times 2$
(ii) s.p → $\frac{dy}{dx} = 0$ → $(2x-3)^2 = 1$
→ $x = 2$ or $x = 1$
If $x = 2$, 2nd diff = +ve → MIN
If $x = 1$, 2nd diff = -ve → MAX

Q12: (i) $\frac{dy}{dx} = 2 - \frac{16}{x^3}$
 $\frac{d^2y}{dx^2} = \frac{48}{x^4}$
(ii) $\frac{dy}{dx} = 0$ → $x = 2$, $y = 6$.
 $\frac{d^2y}{dx^2}$ is +ve Minimum.

Q13: (i) $4x + 2\pi r = 80$
 $A = x^2 + \pi r^2$
→ $A = \frac{(\pi+4)x^2 - 160x + 1600}{\pi}$
(ii) $\frac{dA}{dx} = \frac{2(\pi+4)x - 160}{\pi}$
= 0 when $x = \frac{160}{2(\pi+4)}$ or 11.2

Q14: $y = \frac{12}{x^2 + 3}$
(i) $\frac{dy}{dx} = -12(x^2 + 3)^{-2} \times 2x$
(ii) At $x = 1$, $m = -\frac{3}{2}$
 m of normal = $\frac{2}{3}$
Eqn of normal
 $y - 3 = \frac{2}{3}(x - 1)$
(iii) $\frac{dy}{dx} = \frac{dy}{dx} \times \frac{dx}{dt} = -\frac{3}{2} \times 0.012$
→ -0.018

Q15: $\frac{dy}{dx} = 3\sqrt{x} - 6$ (9, 2)
(i) $y = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - 6x (+c)$
(9, 2) $2 = 54 - 54 + c$
→ $c = 2$.
(ii) $\frac{dy}{dx} = 0$ → $x = 4$
 $\frac{d^2y}{dx^2} = \frac{3x^{-\frac{1}{2}}}{2}$
→ +ve (or $\frac{3}{4}$) Minimum

Q16: (i) $4xh + 2x^2 = 96$
→ $h = \frac{24}{x} - \frac{x}{2}$
 $V = x^2h$ → $V = 24x - \frac{x^3}{2}$
(ii) $\frac{dV}{dx} = 24 - \frac{3x^2}{2}$
= 0 when $x = 4$
→ $V = 64$.
(iii) $\frac{d^2V}{dx^2} = -3x$ → Max.

Q17: $y = \frac{1}{6}(2x-3)^3 - 4x$
(i) $\frac{dy}{dx} = \frac{1}{6} \times 3 \times (2x-3)^2 \times 2 - 4$
(ii) $x = 0$, $y = -\frac{27}{6}$,
 $y + \frac{27}{6} = 5x$ → $2y + 9 = 10x$
(iii) $(2x-3)^2 - 4$ (> 0)
→ $x = 2\frac{1}{2}$ or $\frac{1}{2}$
→ $x > 2\frac{1}{2}$, $x < \frac{1}{2}$.

Q18: (i) $2x + 2y + \frac{\pi x}{2} = 60$
→ $y = 30 - x - \frac{\pi x}{4}$
(ii) $A = xy + \frac{\pi x^2}{4}$
 $= x(30 - x - \frac{\pi x}{4}) + \frac{\pi x^2}{4}$
 $= 30x - x^2$
(iii) $\frac{dA}{dx} = 30 - 2x$
= 0 when $x = 15$ cm
(iv) Max.

Q19:

(i) $(k(2t-1))^{-1/2}$
 $0.7(2t-1)^{-1/2}$

(ii) Sub $t = 5$ into their deriv
0.23(3)

Q20: (i) $h = \frac{8}{x^2}$

$$A = \frac{1}{2}x^2 + 2 \times \frac{1}{2}xh + 2xh + \frac{5}{4}x \times \frac{4}{5}x$$

$$A = (3/2)x^2 + 3xh$$

$$A = \frac{3}{2}x^2 + 3x \times \frac{8}{x^2}$$

$$A = \frac{3}{2}x^2 + \frac{24}{x}$$

(ii) $\frac{dA}{dx} = 3x - \frac{24}{x^2} = 0$

$$x = 2$$

$$\frac{d^2A}{dx^2} = 3 + \frac{48}{x^3}$$

> 0 when $x = 2$ hence minimum

Q21:

(i) $\frac{dy}{dx} = \frac{-1}{(x-3)^2} + 1$

$$\frac{d^2y}{dx^2} = \frac{2}{(x-3)^3}$$

(ii) $(x-3)^2 = 1 \Rightarrow x-3 = \pm 1$

$$x = 4, 2$$

$$y = 5, 1$$

When $x = 4$ $\frac{d^2y}{dx^2} > 0$ ($= 2$) \Rightarrow min

When $x = 2$ $\frac{d^2y}{dx^2} < 0$ ($= -2$) \Rightarrow max

Q22:

$$\left(\frac{dv}{dr}\right) 4\pi r^2$$

$$= 4\pi \times 10^2$$

$$\frac{dr}{dt} = \frac{\frac{dv}{dr}}{\frac{dv}{dr}} \text{ OE used}$$

$$\frac{50}{4\pi \times 10^2} = \frac{1}{8\pi} \text{ or } 0.0398$$

Q23:

(i) $z = 3x + 2\left(\frac{600}{x}\right)$ or $x\left(\frac{z-3x}{2}\right) = 600$ OE

\rightarrow AG

(ii) $\frac{dz}{dx} = 3 - \frac{1200}{x^2}$ or $\frac{dz}{dy} = 2 - \frac{1800}{y^2}$

$$= 0 \rightarrow x = 20$$

$$\text{or } = 0 \rightarrow y = 30$$

$$z = 60 + \frac{120}{20} = 120$$

$$\frac{d^2z}{dx^2} = \frac{2400}{x^3}$$

$> 0 \Rightarrow$ minimum

Q24:

(i) $\frac{3(1+2x)^{-1}}{-1} + (c)$

$$y = \frac{3(1+2x)^{-1}}{-2} + (c)$$

Sub (1, (1/2))

$$\frac{1}{2} = \frac{3}{-2} + c \Rightarrow c = 1$$

(ii) $(1+2x)^2(>)9$ or $4x^2+4x-8(>)0$ OE

$$1, -2$$

$$x > 1, x < -2 \text{ ISW}$$

Q25:

$$y = \frac{4}{3x-4}$$

(i) $\frac{dy}{dx} = -4(3x-4)^{-2} \times 3$

If $x = 2, m = -3$

Eqn of tangent $y - 2 = -3(x - 2)$

(ii) $\tan \theta = \pm(-3)$

$$\rightarrow \theta = \pm 108.4^\circ \text{ (or } \pm 71.6^\circ)$$

or scalar product, $\tan \theta = y\text{-step} \div x\text{-step}$
or use of $\tan(A-B)$ M1A1 for each

Q26:

$$y = \frac{2x^3+5}{x} = 2x^2 + \frac{5}{x}$$

$$d/dx = 4x - \frac{5}{x^2} \text{ or } 4x - 5x^{-2}$$

Q27:

$$\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 1 \quad P(9, 5)$$

(i) $y = 4\sqrt{x} - x + (c)$

Uses (9, 5) in an integrated expression
 $\rightarrow c = 2$

(ii) $\frac{dy}{dx} = 0 \rightarrow x = 4, y = 6$

(iii) $\frac{d^2y}{dx^2} = -x^{-3/2} \rightarrow -ve \rightarrow \text{Max}$

(iv) $\frac{dy}{dx} = -\frac{1}{3}$ Perpendicular $m = 3$
 $\tan \theta = 3$ Angle is $\tan^{-1}3$
 $k = 3$

Q28:

$$\frac{\partial y}{\partial x} = 9x^2 - 12x + 4$$

$$(3x-2)^2 \geq 0$$

Q29:

(i) $y = \frac{1}{6(48-8x)}$ oe

(ii) $A = 4xy + 2xy$ or $3xy + 3xy = 6xy$

$$A = x(48-8x) = 48x - 8x^2$$

(iii) $\frac{\partial A}{\partial x} = 48 - 16x$

$$A = 72 \text{ cao}$$

$$\frac{\partial^2 A}{\partial x^2} = -16 (< 0) \Rightarrow \text{Maximum}$$



Q30:

(i) $f'(3) = 0 \Rightarrow 18 + 3k - 12 = 0$
 $k = -2$
 $(x-3)(x+2) = 0$
 $x = -2$, (Allow also $= 3$)

(ii) $f'(x) = 4x - 2$
 $f'(3) > 0$ hence min at P
 $f'(-2) < 0$ hence max at Q

(iii) $f(x) = \frac{2}{3}x^3 - x^2 - 12x + c$
Sub $(3, -10) \rightarrow -10 = 18 - 9 - 36 + c$
 $c = 17$

Q31: $1000k = 3.2 \Rightarrow k = \frac{3.2}{1000}$ or $\frac{2}{625}$ or 0.0032 oe

$$\left(\frac{dM}{dr}\right) = 3kr^2$$

$$\frac{dM}{dt} = \frac{dM}{dr} \times \frac{dr}{dt} \text{ used e.g. } 3 \times k \times 10^2 \times 0.1$$

0.096